

Admissible operators and \mathcal{H}_∞ calculus

Hans Zwart*

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Abstract

Given a Hilbert space and the generator A of a strongly continuous, exponentially stable, semigroup on this Hilbert space. For any $g(-s) \in \mathcal{H}_\infty$ we show that there exists an infinite-time admissible output operator $g(A)$. If g is rational, then this operator is bounded, and equals the “normal” definition of $g(A)$. In particular, when $g(s) = 1/(s + \alpha)$, $\alpha \in \mathbb{C}_0^+$, then this admissible output operator equals $(\alpha I - A)^{-1}$.

Although in general $g(A)$ may be unbounded, we always have that $g(A)$ multiplied by the semigroup is a bounded operator for every (strictly) positive time instant. Furthermore, when there exists an admissible output operator C such that (C, A) is exactly observable, then $g(A)$ is bounded for all g 's with $g(-s) \in \mathcal{H}_\infty$, i.e., there exists a bounded \mathcal{H}_∞ -calculus. Moreover, we rediscover some well-known classes of generators also having a bounded \mathcal{H}_∞ -calculus.

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1 Introduction

Functional calculus is a sub-field of mathematics with a long history. It started in the thirties of the last century with the work by von Neumann for self-adjoint operators [11], and was further extended by many researchers,

*University of Twente, Faculty of Electrical Engineering, Mathematics and Computer Science, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands, h.j.zwart@math.utwente.nl

see e.g. [8] and [3]. For an overview, see the book by Markus Haase, [7]. The basic idea behind functional calculus for the operator A is to construct a mapping from an algebra of (scalar) functions to the class of (bounded) operators, such that

- The function identically equals to one is mapped to the identity operator;
- If $f(s) = (s - a)^{-1}$, then $f(A) = (sI - A)^{-1}$;
- Furthermore, the operator associated to $f_1 \cdot f_2$ equals $f(A)f_2(A)$.

Before we explain the contribution of this paper, we introduce some notation. By X we denote separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and by A we denote an unbounded operator from its domain $D(A) \subset X$ to X . We assume that A generates an exponentially stable semigroup on X , which we denote by $(T(t))_{t \geq 0}$.

By \mathcal{H}_∞^- we denote the space of all bounded, analytic functions defined on the half-plane $\mathbb{C}^- := \{s \in \mathbb{C} \mid \operatorname{Re}(s) < 0\}$. It is clear that this function class is an algebra under pointwise multiplication and addition. Hence this could serve as a class for which one could build a functional calculus. However, it is known that there exists a generator of exponential stable semigroup, which does not have a functional calculus with respect to \mathcal{H}_∞^- . For proof of this and many more we refer to [1], [7], and the references therein. Although a bounded functional calculus is not possible, an unbounded functional calculus is always possible.

Theorem 1.1 *Under the assumptions stated above, we have that for all $g \in \mathcal{H}_\infty^-$ there exists an operator $g(A)$ which is bounded from the domain of A to X , and which is admissible, i.e.,*

$$\int_0^\infty \|g(A)T(t)x_0\|^2 dt \leq \gamma_A \|g\|_\infty^2 \|x_0\|^2, \quad x_0 \in X.$$

The mapping $g \mapsto g(A)$ satisfies the conditions of a functional calculus. Furthermore, for all $t > 0$, we have that $g(A)T(t)$ can be extended to a bounded operator, and

$$\|g(A)T(t)\| \leq \frac{\gamma}{\sqrt{t}}.$$

Apart from proving this theorem, we shall also rediscover some classes of generators for which $g(A)$ is bounded for all $g \in \mathcal{H}_\infty^-$, i.e., for which there is a bounded functional calculus.

For the proof of the above result, we need beside the Hardy space \mathcal{H}_∞^- also the Hardy spaces $\mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$.

$\mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$ denote the Laplace transform, \mathcal{L} , of functions in $L^2((0, \infty), X)$ and $L^2((-\infty, 0), X)$, respectively. It is known that this transformation is an isometry. Every function in \mathcal{H}_∞^- , $\mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$ has a unique extension to the imaginary axis on which this functions are bounded, and square integrable, respectively. Furthermore, the norm of $g \in \mathcal{H}_\infty^-$ equals the (essential) supremum over the imaginary axis of the boundary function. Let $f(t)$ be a function in $L^2((0, \infty), X)$ with Laplace transform $F(s)$, and let $f_{\text{ext}}(t)$ be the function in $L^2((-\infty, \infty), X)$ defined by

$$f_{\text{ext}}(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then the Fourier transform \hat{f}_{ext} of $f_{\text{ext}}(t)$ satisfies $\hat{f}_{\text{ext}}(\omega) = F(i\omega)$, for almost all $\omega \in \mathbb{R}$. Here $F(i\cdot)$ denote the boundary function of the Laplace transform $F(s)$.

We define the following Toeplitz operator on $L^2((0, \infty); X)$

Definition 1.2 *Let g be an element of \mathcal{H}_∞^- . Associated to this function we define the mapping M_g as*

$$M_g f = \mathcal{L}^{-1}(\Pi(gF)), \quad f \in L^2((0, \infty), X), \quad (1)$$

where F denotes the Laplace transform of f . Π denotes the projection onto $\mathcal{H}_2(X)$.

It is clear that this is a linear bounded map from $L^2((0, \infty); X)$ into itself, and

$$\|M_g\| \leq \|g\|_\infty. \quad (2)$$

Furthermore, it follows easily from (1) that if K is a bounded mapping on X , then it commutes with M_g , i.e.,

$$KM_g = M_g K. \quad (3)$$

It is easy to see that \mathcal{H}_∞^- is an algebra under the multiplication and addition. In particular $g_1 g_2 \in \mathcal{H}_\infty^-$ whenever $g_1, g_2 \in \mathcal{H}_\infty^-$. Furthermore, we have the following result.

Lemma 1.3 *Let g_1 and g_2 be elements of \mathcal{H}_∞^- . Then*

$$M_{g_1 g_2} = M_{g_1} M_{g_2}. \quad (4)$$

In particular, if g is invertible in \mathcal{H}_∞^- , then M_g is (boundedly) invertible and $(M_g)^{-1} = M_{g^{-1}}$.

Proof We use the fact that any $g \in \mathcal{H}_\infty^-$ maps \mathcal{H}_2^\perp into \mathcal{H}_2^\perp .

$$\begin{aligned} M_{g_1} M_{g_2} f &= \mathcal{L}^{-1} (\Pi g_1 (\Pi (g_2 F))) \\ &= \mathcal{L}^{-1} (\Pi (g_1 g_2 F)) + \mathcal{L}^{-1} (\Pi (g_1 (I - \Pi) (g_2 F))) \\ &= \mathcal{L}^{-1} (\Pi (g_1 g_2 F)) + 0, \end{aligned}$$

where we have used the above mentioned fact that $g_1(I - \Pi)$ maps into \mathcal{H}_2^\perp , and so $\Pi g_1(I - \Pi) = 0$. Since by definition $\mathcal{L}^{-1} (\Pi (g_1 g_2 F))$ equals $M_{g_1 g_2} f$, we have proved the first assertion.

The last assertion follows directly, since $M_1 = I$. \square

By σ_τ we denote the shift with $\tau \geq 0$, i.e.,

$$(\sigma_\tau(f))(t) = f(t + \tau), \quad t \geq 0. \quad (5)$$

This is also a linear bounded map from $L^2((0, \infty); X)$ into itself. This mapping commutes with M_g as is shown next.

Lemma 1.4 *For all $\tau > 0$ and all g in \mathcal{H}_∞^- , we have that*

$$\sigma_\tau (M_g f) = M_g (\sigma_\tau f), \quad f \in L^2((0, \infty), X). \quad (6)$$

Proof We use the following well-known equality. If h is Fourier transformable, then the Fourier transform of $h(\cdot + \tau)$ equals $e^{i\omega\tau} \hat{h}(\omega)$, where \hat{h} denotes the Fourier transform of h .

Let $h \in L^2((0, \infty); X)$, then

$$\mathcal{L}(\sigma_\tau h) = (\widehat{\sigma_\tau h})_{\text{ext}} = \widehat{\sigma_\tau h_{\text{ext}}} - \hat{q} = e^{i\omega\tau} \widehat{h_{\text{ext}}} - \hat{q} = e^{i\omega\tau} \mathcal{L}(h) - \hat{q}, \quad (7)$$

with $q \in L^2((-\infty, 0); X)$. In particular, we find for every $h \in L^2(0, \infty); X$ that

$$\mathcal{L}(\sigma_\tau h) = \Pi (\mathcal{L}(\sigma_\tau h)) = \Pi (e^{i\omega\tau} \mathcal{L}(h)) - 0 = \mathcal{L} (M_{e^{i\cdot\tau}} h), \quad (8)$$

where we have used that $e^{i\omega\tau}$ is the boundary function corresponding to $e^{is\tau} \in \mathcal{H}_\infty^-$.

Using (7) we see that

$$M_g(\sigma_\tau f) = \mathcal{L}^{-1}(\Pi(g e^{i\tau} \mathcal{L}(f))) - \mathcal{L}^{-1}(\Pi(g \hat{q})) = \mathcal{L}^{-1}(\Pi(g e^{i\tau} \mathcal{L}(f))), \quad (9)$$

since $\hat{q} \in \mathcal{H}_2^\perp(X)$, and since $g \in \mathcal{H}_\infty^-$. Using Lemma 1.3, we find that

$$M_g(\sigma_\tau f) = \mathcal{L}^{-1}(\Pi(g e^{i\tau} \mathcal{L}(f))) = M_{e^{i\cdot\tau}g}f = M_{e^{i\cdot\tau}}M_gf. \quad (10)$$

Now using (8), we see that

$$M_g(\sigma_\tau f) = \sigma_\tau(M_gf). \quad (11)$$

□

2 Output maps and admissible output operators

In this section we study admissible operators which commute with the semigroup. We begin by defining well-posed output maps.

Definition 2.1 *Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on the Hilbert space X , and let Y be another Hilbert space. We say that the mapping \mathcal{O} is a well-posed (infinite-time) output map if*

- \mathcal{O} is a bounded linear mapping from X into $L^2((0, \infty); Y)$, and
- For all $\tau \geq 0$ and all $x_0 \in X$, we have that $\sigma_\tau \mathcal{O}x_0 = \mathcal{O}(T(\tau)x_0)$.

Closely related to well-posed output mappings are admissible operators, which are defined next.

Definition 2.2 *Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on the Hilbert space X . Let $D(A)$ be the domain of its generator A . A linear mapping C from $D(A)$ to Y , another Hilbert space, is said to be an (infinite-time) admissible output operator for $(T(t))_{t \geq 0}$ if $CT(\cdot)x_0 \in L^2((0, \infty), Y)$ for all $x_0 \in D(A)$ and there exists an m independent of x_0 such that*

$$\int_0^\infty \|CT(t)x_0\|_Y^2 dt \leq m \|x_0\|_X^2. \quad (12)$$

If C is (infinite-time) admissible, then for all $x_0 \in X$ we can uniquely define an $L^2((0, \infty), X)$ -function. We denote this function by $CT(\cdot)x_0$. Hence $\mathcal{O} : X \rightarrow L^2((0, \infty); Y)$ defined by $\mathcal{O}x_0 = CT(\cdot)x_0$ is a well-posed output map. From [12] we know that the converse holds as well.

Lemma 2.3 *If \mathcal{O} is a well-posed output mapping, then there exists a (unique) linear bounded mapping from $D(A)$ to Y , C , such that $\mathcal{O}x_0 = CT(\cdot)x_0$ for all x_0 .*

In the sequel of this section we concentrate on admissible output operators which commute with the semigroup, i.e., C a linear operator from $D(A)$ to X and

$$CT(t)x_0 = T(t)Cx_0 \quad \text{for all } t \geq 0 \text{ and } x_0 \in D(A). \quad (13)$$

For these operators we have the following results.

Lemma 2.4 *Let C be the admissible output operator associated with the well-posed output map \mathcal{O} . Then (13) holds if and only if for all $t \geq 0$ there holds $\mathcal{O}T(t) = T(t)\mathcal{O}$.*

Theorem 2.5 *Let C be a bounded linear operator from $D(A)$ to X , which is admissible for the exponentially stable semigroup $(T(t))_{t \geq 0}$ and which commutes with this semigroup. Then the following holds*

1. *For all $x_0 \in D(A)$, we have that $CA^{-1}x_0 = A^{-1}Cx_0$.*
2. *For all $t > 0$, the operator $CT(t) : D(A) \rightarrow X$ can be extended to a bounded operator on X . Furthermore, $\|CT(t)\| \leq \gamma t^{-1/2}$ for some γ independent of t .*

Proof The first assertion follows easily from (13) by using Laplace transforms. We concentrate on the second assertion.

Let $x_0 \in D(A)$ and $x_1 \in X$, then for $t > 0$ we have that

$$\begin{aligned}
t\langle x_1, CT(t)x_0 \rangle &= \int_0^t \langle x_1, CT(\tau)x_0 \rangle d\tau \\
&= \int_0^t \langle x_1, CT(\tau)T(t-\tau)x_0 \rangle d\tau \\
&= \int_0^t \langle x_1, T(\tau)CT(t-\tau)x_0 \rangle d\tau \\
&= \int_0^t \langle T(\tau)^*x_1, CT(t-\tau)x_0 \rangle d\tau \\
&\leq \sqrt{\int_0^t \|T(\tau)^*x_1\|^2 d\tau} \sqrt{\int_0^t \|CT(t-\tau)x_0\|^2 d\tau}.
\end{aligned}$$

Using the fact that the semigroup, and hence its adjoint, are uniformly bounded, and the fact that C is (infinite-time) admissible, we find that

$$t\langle x_1, CT(t)x_0 \rangle \leq \sqrt{t}M\|x_1\|m\|x_0\|.$$

Since this holds for all $x_1 \in X$, we conclude that

$$t\|CT(t)x_0\| \leq \sqrt{t}mM\|x_0\|.$$

This inequality holds for all $x_0 \in D(A)$. The domain of a generator is dense, and hence we have proved the second assertion. \square

From Theorem 2.5 it is clear that if the semigroup is surjective, then any admissible C which commutes with the semigroup is bounded. However, this does not hold for a general semigroup as is shown in the following example. Furthermore, this example also shows that the estimate in the previous theorem cannot be improved.

Example 2.6 Let $\{\phi_n, n \in \mathbb{N}\}$ be an orthonormal basis of X , and define for $t \geq 0$ the operator

$$T(t) \sum_{n=1}^N \alpha_n \phi_n = \sum_{n=1}^N e^{-n^2 t} \alpha_n \phi_n. \quad (14)$$

It is not hard to show that this defines an exponentially stable C_0 -semigroup on X . The infinitesimal generator A is given by

$$A \sum_{n=1}^N \alpha_n \phi_n = \sum_{n=1}^N -n^2 \alpha_n \phi_n.$$

with domain

$$D(A) = \{x = \sum_{n=1}^{\infty} \alpha_n \phi_n \in X \mid \sum_{n=1}^{\infty} |n^2 \alpha_n|^2 < \infty\}$$

We define C as the square root of $-A$, i.e.

$$C \sum_{n=1}^N \alpha_n \phi_n = \sum_{n=1}^N n \alpha_n \phi_n \quad (15)$$

with domain

$$D(C) = \{x = \sum_{n=1}^{\infty} \alpha_n \phi_n \in X \mid \sum_{n=1}^{\infty} |n \alpha_n|^2 < \infty\}.$$

A straightforward calculation gives that for $x_0 = \sum_{n=1}^N \alpha_n \phi_n$, we have that

$$\int_0^{\infty} \|CT(t)x_0\|^2 dt = \int_0^{\infty} \sum_{n=1}^N |ne^{-n^2 t} \alpha_n|^2 dt = \frac{1}{2} \sum_{n=1}^N |\alpha_n|^2 = \frac{1}{2} \|x_0\|^2.$$

Since the finite sums lie dense, we conclude that C is admissible. It is easy to see that C commutes with the semigroup, and thus from Theorem 2.5 we have that

$$\|CT(t)\| \leq \frac{\gamma}{\sqrt{t}}. \quad (16)$$

for some γ independent of t .

Next choose $x_0 = \phi_n$ and $t = n^{-2}$. Using (14) and (15) we see that

$$CT(t)x_0 = ne^{-1} \phi_n = \frac{e^{-1}}{\sqrt{t}} x_0,$$

and thus the estimate (16) cannot be improved.

The Lebesgue extension of an admissible operator is defined by

$$C_L x = \lim_{t \rightarrow 0} \frac{1}{t} C \int_0^t T(\tau) x d\tau,$$

where

$$D(C_L) = \{x \in X \mid \text{limit exists}\}.$$

A similar extension can be define using the resolvent. The Lambda extension of an admissible operator is defined by

$$C_\Lambda x = \lim_{\lambda \rightarrow \infty} \lambda C(\lambda I - A)^{-1}x,$$

where

$$D(C_\Lambda) = \{x \in X \mid \text{limit exists}\}.$$

The relation between these extension is still not completely understood, but for admissible operators which commute with the semigroup, we have that both extensions are closed operators.

Lemma 2.7 *Let C be an admissible operator which commutes with the semigroup, then the same holds for its Lebesgue and Lambda extension. Furthermore, these extensions are closed operators.*

Proof Since A^{-1} and CA^{-1} are bounded, we find for $x_0 \in D(C_L)$

$$\begin{aligned} A^{-1}C_L x_0 &= A^{-1} \lim_{t \downarrow 0} \frac{1}{t} C \int_0^t T(\tau) x_0 d\tau = \lim_{t \downarrow 0} \frac{1}{t} A^{-1} C \int_0^t T(\tau) x_0 d\tau \\ &= \lim_{t \downarrow 0} \frac{1}{t} C A^{-1} \int_0^t T(\tau) x_0 d\tau = C A^{-1} \lim_{t \downarrow 0} \frac{1}{t} \int_0^t T(\tau) x_0 d\tau \\ &= C A^{-1} x_0 = C_L A^{-1} x_0, \end{aligned}$$

where we have used that $\int_0^t T(\tau) x_0 d\tau \in D(A)$ and C commutes with A^{-1} . This proves the first assertion.

Using once more that CA^{-1} and A^{-1} are bounded, we have for $x_0 \in D(C_L)$

$$\begin{aligned} C A^{-1} \int_0^t T(\tau) x_0 d\tau &= \int_0^t C A^{-1} T(\tau) x_0 d\tau \\ &= \int_0^t T(\tau) C A^{-1} x_0 d\tau \\ &= \int_0^t T(\tau) A^{-1} C_L x_0 d\tau = A^{-1} \int_0^t T(\tau) C_L x_0 d\tau. \end{aligned}$$

Let x_n be a sequence in $D(C_L)$ which converges to $x \in X$, such that $C_L x_n$ converges to $z \in X$. Then by the above we find that

$$C A^{-1} \int_0^t T(\tau) x d\tau = A^{-1} \int_0^t T(\tau) z d\tau \quad (17)$$

Since $\int_0^t T(\tau)x d\tau \in D(A)$, we find that

$$A^{-1} \int_0^t T(\tau)z d\tau = CA^{-1} \int_0^t T(\tau)x d\tau = A^{-1}C \int_0^t T(\tau)x d\tau. \quad (18)$$

Hence we have that

$$\int_0^t T(\tau)z d\tau = C \int_0^t T(\tau)x d\tau.$$

Since $t^{-1} \int_0^t T(\tau)z d\tau$ converges to z for $t \downarrow 0$, we conclude from the above equality that $x \in D(C_L)$ and $C_L x = z$.

The proof for C_Λ goes very similarly. Basically in the above proof, $\int_0^t T(\tau)x d\tau$ is replaced by $(\lambda I - A)^{-1}x$. \square

By Weiss [14] we have that C_Λ is an extension of C_L . We claim that for admissible C 's which commute with the semigroup they are equal.

3 \mathcal{H}_∞ -calculus

For $g \in \mathcal{H}_\infty^-$ we define the following mapping from X to $L^2((0, \infty); X)$

$$\mathfrak{D}_g x_0 = M_g (T(t)x_0). \quad (19)$$

Hence we have taken in Definition 1.2 $f(t) = T(t)x_0$.

It is clear that \mathfrak{D}_g is a linear bounded operator from X into $L^2((0, \infty); X)$. Furthermore, from (6) we have that

$$\sigma_\tau (\mathfrak{D}_g x_0) = M_g (\sigma_\tau (T(t)x_0)) = M_g T(t + \tau)x_0 = \mathfrak{D}_g (T(\tau)x_0), \quad (20)$$

where we have used the semigroup property. Hence \mathfrak{D}_g is a well-posed output map, and so by Lemma 2.3 we conclude that \mathfrak{D}_g can be written as

$$\mathfrak{D}_g x_0 = g(A)T(t)x_0 \quad (21)$$

for some infinite-time admissible operator $g(A)$ which is bounded from the domain of A to X .

Since for all $t, \tau \in [0, \infty)$ there holds $T(\tau)T(t) = T(t)T(\tau)$, we conclude from (19) and (3) that

$$\mathfrak{D}_g T(t) = T(t)\mathfrak{D}_g, \quad t \geq 0.$$

Hence by (21), we see that $g(A)$ is an admissible operator which commutes with the semigroup. Theorem 2.5 implies that for $t > 0$, $g(A)T(t)$ can be extended to a bounded operator and

$$\|g(A)T(t)\| \leq \frac{\gamma}{\sqrt{t}}. \quad (22)$$

Note that for $t \in [0, 1]$ this γ can be chosen as $\sup_{t \in [0, 1]} \|T(t)\| \cdot \|g\|_\infty$.

The Laplace transform of \mathfrak{D}_g equals $g(A)(sI - A)^{-1}$. Combining this with the definition of \mathfrak{D}_g , implies that

$$\|g(A)(sI - A)^{-1}\| \leq \frac{\|g\|_\infty}{\sqrt{\operatorname{Re}(s)}} \|x_0\|, \quad (23)$$

where we have taken the norm in X , see also Weiss [13].

Since we have written this admissible operator as the function g working on the operator A , there is likely to be a relation with functional calculus. This is shown next.

Lemma 3.1 *If $g \in \mathcal{H}_\infty^-$ is the inverse Fourier transform of the function h , with $h \in L^1(-\infty, \infty)$ with support in $(-\infty, 0)$, then $g(A)$ is bounded*

$$g(A)x_0 = \int_0^\infty T(t)h(-t)x_0 dt, \quad (24)$$

and so $g(A)$ corresponds to the classical definition of the function of an operator.

So if g is the Fourier transform of an absolutely integrable function, then $g(A)$ is bounded. We would like to know when it is bounded for every g . For this, we extend the definition of \mathfrak{D}_g .

Let C be an admissible output operator for the semigroup $(T(t))_{t \geq 0}$. By definition, we know that $CT(\cdot)x_0 \in L^2((0, \infty); Y)$ for all $x_0 \in X$. We define

$$(C \circ \mathfrak{D}_g)x_0 = M_g(CT(t)x_0) \quad (25)$$

It is clear that this is a bounded mapping from X to $L^2((0, \infty); Y)$.

As before we have that

$$\sigma_\tau((C \circ \mathfrak{D}_g)(x_0)) = (C \circ \mathfrak{D}_g)(T(\tau)x_0). \quad (26)$$

And so we can write $(C \circ \mathfrak{D}_g)x_0$ as $\tilde{C}_g T(\cdot)x_0$ for some infinite-time admissible \tilde{C}_g . We have that

Lemma 3.2 *The infinite-time admissible operator \tilde{C}_g satisfies*

$$\tilde{C}_g x_0 = Cg(A)x_0, \quad \text{for } x_0 \in D(A^2). \quad (27)$$

Proof For $x_0 \in D(A^2)$, we introduce $x_1 = Ax_0$. Then the following equalities hold in $L^2((0, \infty); Y)$.

$$\begin{aligned} \tilde{C}_g T(t)x_0 &= (C \circ \mathfrak{D}_g) x_0 \\ &= M_g (CT(t)x_0) \\ &= M_g (CT(t)A^{-1}x_1) \\ &= M_g (CA^{-1}T(t)x_1) \\ &= CA^{-1}g(A)T(t)x_1 \\ &= Cg(A)T(t)A^{-1}x_1 = Cg(A)T(t)x_0, \end{aligned}$$

where we have used (3). Since both functions are continuous at zero, we find that (27) holds. \square

Based on this result, we denote \tilde{C}_g by $C \circ g(A)$.

Using this, we can prove the following theorems.

Theorem 3.3 *The mapping $g \mapsto g(A)$ forms a (unbounded) \mathcal{H}_∞^- -calculus.*

Proof It only remains to show that $(g_1 g_2)(A) = g_1(A)g_2(A)$. By Lemma 1.3 we have that

$$\mathfrak{D}_{g_1 g_2} x_0 = M_{g_1 g_2} (T(t)x_0) = M_{g_1} M_{g_2} (T(t)x_0).$$

For $x_0 \in D(A)$ the last expression equals $M_{g_1} (g_2(A)T(t)x_0)$, see (21). Since $g_2(A)$ commutes with the semigroup, we find that

$$\mathfrak{D}_{g_1 g_2} x_0 = M_{g_1} (T(t)g_2(A)x_0).$$

Using (21) twice, we obtain

$$(g_1 g_2)(A)T(t)x_0 = \mathfrak{D}_{g_1 g_2} x_0 = g_1(A)T(t)g_2(A)x_0$$

This is an equality in $L^2((0, \infty); X)$. However, if we take $x_0 \in D(A^2)$, then this holds point-wise, and so for $x_0 \in D(A^2)$.

$$(g_1 g_2)(A)x_0 = g_1(A)g_2(A)x_0$$

This concludes the proof. \square

Theorem 3.4 *If there exists an admissible C such that (C, A) is exactly observable, i.e., there exists an $m_1 > 0$ such that for all $x_0 \in X$ there holds*

$$\int_0^\infty \|CT(t)x_0\|^2 dt \geq m_1 \|x_0\|^2$$

then $g(A)$ is bounded for every $g \in \mathcal{H}_\infty^-$. Furthermore, if m_2 is the admissibility constant, see equation (12), then

$$\|g(A)\| \leq \sqrt{\frac{m_2}{m_1}} \|g\|_\infty. \quad (28)$$

Proof Let $x_0 \in D(A^2)$

$$\begin{aligned} m_1 \|g(A)x_0\|^2 &\leq \|CT(t)g(A)x_0\|_{L^2((0,\infty);Y)}^2 \\ &= \|Cg(A)T(t)x_0\|_{L^2((0,\infty);Y)}^2 \\ &= \|C \circ \mathfrak{D}_g x_0\|_{L^2((0,\infty);Y)}^2 \\ &\leq \|g\|_\infty^2 \|CT(t)x_0\|_{L^2((0,\infty);Y)}^2 \\ &\leq m_2 \|g\|_\infty^2 \|x_0\|^2. \end{aligned}$$

Since $D(A^2)$ is dense, we obtain the result. \square

As a corollary we obtain the well-known von Neumann inequality. Recall that the operator A is dissipative if

$$\langle x_0, Ax_0 \rangle + \langle Ax_0, x_0 \rangle \leq 0 \quad \text{for all } x_0 \in D(A). \quad (29)$$

Corollary 3.5 *If A is a dissipative operator and its corresponding semigroup is exponentially stable, then A has a bounded \mathcal{H}_∞^- calculus and for all $g \in \mathcal{H}_\infty^-$*

$$\|g(A)\| \leq \|g\|_\infty. \quad (30)$$

Proof Since A is dissipative and since its semigroups is exponentially stable, we have that A^{-1} is bounded and dissipative. We define Q via

$$\langle x_1, Qx_2 \rangle = -\langle A^{-1}x_1, x_2 \rangle - \langle x_1, A^{-1}x_2 \rangle, \quad x_1, x_2 \in X. \quad (31)$$

It is easy to see that Q is bounded, self-adjoint and by the dissipativity of A^{-1} we have that $Q \geq 0$. Define on the domain of A the operator C as $C = \sqrt{Q}A$, then from (31) we find that

$$-\langle Cx_1, Cx_2 \rangle = \langle x_1, Ax_2 \rangle + \langle Ax_1, x_2 \rangle, \quad x_1, x_2 \in D(A). \quad (32)$$

Combining this Lyapunov equation with the exponential stability, gives that for all $x_0 \in D(A)$

$$\int_0^\infty \|CT(t)x_0\|^2 dt = \|x_0\|^2. \quad (33)$$

Thus we see that the constants m_1 and m_2 in Theorem 3.4 can be chosen to be one, and so (28) gives the results. \square

If A generates an exponentially stable semigroup and if there exists an admissible C for which (C, A) is exactly observable, then it is not hard to show that the semigroup is similar to a contraction semigroup. Using this, one can also obtain the above result by Theorem G of [1]. The following result has been proved by McIntosh in [10].

Theorem 3.6 *Assume that A generates an exponentially stable semigroup. If $(-A)^{\frac{1}{2}}$ is admissible for $(T(t))_{t \geq 0}$ and $(-A^*)^{\frac{1}{2}}$ is admissible for the adjoint semigroup $(T(t)^*)_{t \geq 0}$, then $g(A)$ is bounded for every $g \in \mathcal{H}_\infty^-$. Thus this semigroup has a bounded \mathcal{H}_∞^- -calculus.*

Proof Since $A^{1/2}$ is admissible, Lemma 3.2 gives that $A^{1/2} \circ g(A)$ is also

admissible. Consider for $x_1 \in D(A^*)$ and $x_0 \in D(A^2)$ the following

$$\begin{aligned}
& \langle x_1, g(A)x_0 \rangle - \langle x_1, g(A)T(t)x_0 \rangle \\
&= \int_0^t \langle x_1, (-A)T(\tau)g(A)x_0 \rangle d\tau \\
&= \int_0^t \langle (-A^*)^{\frac{1}{2}}x_1, (-A)^{\frac{1}{2}}g(A)T(\tau)x_0 \rangle d\tau \\
&= \int_0^t \langle (-A^*)^{\frac{1}{2}}T(\frac{\tau}{2})^*x_1, g(A)(-A)^{\frac{1}{2}}T(\frac{\tau}{2})x_0 \rangle d\tau \\
&\leq \sqrt{\int_0^t \|(-A^*)^{\frac{1}{2}}T(\frac{\tau}{2})^*x_1\|^2 d\tau} \sqrt{\int_0^t \|g(A)(-A)^{\frac{1}{2}}T(\frac{\tau}{2})x_0\|^2 d\tau} \\
&\leq \sqrt{\int_0^t \|(-A^*)^{\frac{1}{2}}T(\frac{\tau}{2})^*x_1\|^2 d\tau} \|g\|_\infty \sqrt{\int_0^\infty \|(-A)^{\frac{1}{2}}T(\frac{\tau}{2})x_0\|^2 d\tau} \\
&\leq m_1 \|x_1\| m_2 \|g\|_\infty \|x_0\|,
\end{aligned}$$

where m_1 and m_0 are the admissibility constant of $(-A^*)^{\frac{1}{2}}$ and $(-A)^{\frac{1}{2}}$, respectively. Furthermore, we used (2).

Since the sets $D(A^*)$ and $D(A^2)$ are dense in X , we obtain that

$$\|g(A)\| \leq m_1 m_2 \|g\|_\infty + \|g(A)T(t)\|. \quad (34)$$

By Theorem 2.5 we know that $g(A)T(t)$ is bounded, and so we conclude that $(T(t))_{t \geq 0}$ has a bounded \mathcal{H}_∞^- -calculus. \square

In McIntosh [10] the above theorem was proved using square function estimates. The admissibility of $(-A)^{\frac{1}{2}}$ can be written as

$$\begin{aligned}
m \|x_0\|^2 &\geq \int_0^\infty \|(-A)^{\frac{1}{2}}T(t)x_0\|^2 dt \\
&= \int_0^\infty \|(-tA)^{\frac{1}{2}}T(t)x_0\|^2 \frac{dt}{t}.
\end{aligned}$$

The latter is the “square function estimate” for $\psi(s) = (-s)^{\frac{1}{2}}e^s$, and so the admissibility condition can be seen as a square function estimate. The other condition used in [10] is that the operator A is sectorial on a sector larger than the sector on which the scalar functions are defined. Since we have as function class \mathcal{H}_∞^- and since our operators A are assumed to generate an

exponential semigroup, this condition seems not to be satisfied. However, the admissibility assumptions made in the theorem imply that A generates a bounded analytic semigroup, and so the condition of McIntosh is satisfied.

Lemma 3.7 *Let A generate an exponentially stable semigroup and let $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ be admissible operators for $(T(t))_{t \geq 0}$ and $(T(t)^*)_{t \geq 0}$, respectively. Then A generates a bounded analytic semigroup.*

Proof The proof is similar to the proof of Theorem 2.5. Let $x_1 \in D(A^*)$ and $x_0 \in D(A)$. Then for $t > 0$ we find

$$\begin{aligned}
t \langle x_1, AT(t)x_0 \rangle &= \int_0^t \langle x_1, AT(\tau)x_0 \rangle d\tau \\
&= - \int_0^t \langle (-A^*)^{\frac{1}{2}} x_1, (-A)^{\frac{1}{2}} T(\tau)x_0 \rangle d\tau \\
&= - \int_0^t \langle (-A^*)^{\frac{1}{2}} T(\tau)^* x_1, (-A)^{\frac{1}{2}} T(t-\tau)x_0 \rangle d\tau \\
&\leq \sqrt{\int_0^t \| (-A^*)^{\frac{1}{2}} T(\tau)^* x_1 \|^2 d\tau} \sqrt{\int_0^t \| (-A)^{\frac{1}{2}} T(t-\tau)x_0 \|^2 d\tau} \\
&\leq m_1 \|x_1\| m_2 \|x_0\|,
\end{aligned}$$

where we used that $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ are admissible. Since the domain of A^* and A are dense, we obtain that

$$\|AT(t)\| \leq \frac{M}{t}, \quad t > 0$$

By Theorem II.4.6 of [4], we conclude that A generates a bounded analytic semigroup. \square

From [10] we know that if the conditions of Theorem 3.6 hold, then is the semigroup similar to a contraction (or $(-A)^{\frac{1}{2}}$ is exactly observable). We show this next.

Lemma 3.8 *Under the condition of Theorem 3.6 we have that $(-A)^{\frac{1}{2}}$ is exactly observable, and thus $(T(t))_{t \geq 0}$ is similar to a contraction.*

Proof In idea the proof is the same as that of Theorem 3.6. Let $x_1 \in D(A^*)$ and $x_0 \in D(A)$ We have that

$$\begin{aligned}\langle x_1, x_0 \rangle &= \int_0^\infty \langle x_1, (-A)T(\tau)x_0 \rangle d\tau \\ &= \int_0^\infty \langle (-A^*)^{\frac{1}{2}} x_1, (-A)^{\frac{1}{2}} T(\tau)x_0 \rangle d\tau \\ &= \int_0^\infty \langle (-A^*)^{\frac{1}{2}} T(\frac{\tau}{2})^* x_1, (-A)^{\frac{1}{2}} T(\frac{\tau}{2})x_0 \rangle d\tau.\end{aligned}\quad (35)$$

Hence

$$\begin{aligned}|\langle x_1, x_0 \rangle| &\leq \sqrt{\int_0^\infty \|(-A^*)^{\frac{1}{2}} T(\frac{\tau}{2})^* x_1\|^2 d\tau} \sqrt{\int_0^\infty \|(-A)^{\frac{1}{2}} T(\frac{\tau}{2})x_0\|^2 d\tau} \\ &\leq m_1 \|x_1\| \sqrt{\int_0^\infty \|(-A)^{\frac{1}{2}} T(\frac{\tau}{2})x_0\|^2 d\tau}\end{aligned}$$

Since the domain of A^* is dense we conclude that

$$\|x_0\| = \sup_{x_1 \neq 0} \frac{|\langle x_1, x_0 \rangle|}{\|x_1\|} \leq m_1 \sqrt{\int_0^\infty \|(-A)^{\frac{1}{2}} T(\frac{\tau}{2})x_0\|^2 d\tau}.\quad (36)$$

Thus $(-A)^{\frac{1}{2}}$ is exactly observable. \square

We remark that with the above result, Theorem 3.6 follows also from Theorem 3.4. However, we decided to present this independent proof.

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